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Unitarily Inequivalent Representations in Algebraic Quantum Theory

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It has been maintained that the physical content of a model of a system is completely contained in the C^*-algebra of quasi-local observables $\mathcal{A}$ that is associated with the system. The reason given for this is that the unitarily inequivalent representations of $\mathcal{A}$ are physically equivalent. But, this view is dubious for at least two reasons. First, it is not clear why the physical content does not extend to the elements of the von Neumann algebras that are generated by representations of $\mathcal{A}$. It is shown here that although the unitarily inequivalent representations of $\mathcal{A}$ are physically equivalent, the extended representations are not. Second, this view detracts from special global features of physical systems such as temperature and chemical potential by effectively relegating them to the status of fixed parameters. It is desirable to characterize such observables theoretically as elements of the algebra that is associated with a system rather than as parameters, and thereby give a uniform treatment to all observables. This can be accomplished by going to larger algebras. One such algebra is the universal enveloping von Neumann algebra, which is generated by the universal representation of $\mathcal{A}$; another is the direct integral of factor representations that are associated with the set of values of the global features. Placing interpretive significance on the von Neumann algebras mentioned earlier sheds light on the significance of unitarily inequivalent representations of $\mathcal{A}$, and it serves to show the limitations of the notion of physical equivalence.

KEY WORDS: C^*-algebra; inequivalent representations; von Neumann algebras; algebraic field theory; physical equivalence; foundations of physics.

1. INTRODUCTION

During the mid-to-late 1920s there were two competing versions of non-relativistic, spinless quantum mechanics: Schrödinger's wave mechanics and Heisenberg's matrix mechanics. Through the work of Schrödinger, Dirac, and most importantly von Neumann, it was realized that the two theories are unitarily

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equivalent. This equivalence may be expressed by saying that each of them is a representation of the Heisenberg form of the canonical commutation relations (CCRs):

$$[P_i, Q_j] = P_i Q_j - Q_j P_i = -i\hbar\delta_{ij}$$
$$[P_i, P_j] = [Q_i, Q_j] = 0$$

The indices i, j correspond to degrees of freedom of a system. In what follows, it will suffice to consider just 1 degree of freedom, since the generalization to n-degrees of freedom ($n \in \mathbb{N}$) is straightforward. In Schrödinger's representation, P corresponds to the differential operator $-i\hbar\, d/dx$ and Q to multiplication by the variable x, both acting on elements of $\mathcal{L}^2(\mathbb{R})$ (the space of Lebesgue square integrable functions on the set of real numbers). In Heisenberg's representation, Born and Jordan found formal matrices with infinitely many entries to represent the operators P and Q that acted on the complete vector space over $\mathbb{C}$ of square-summable complex sequences.

There are problems with the operators associated with P and Q above. They arise because these operators are unbounded. As a result, they are only defined on a merely dense subset of the Hilbert space, meaning that they are essentially ambiguous until a domain is specified. The initial and boundary conditions determine the domain, but it is often a difficult matter to specify the domain given those conditions. Moreover, each time such operators are combined algebraically, one must further restrict them to a common dense domain. Finally, it might be thought that there could be realizations of the CCRs as bounded operators on Hilbert space, but it can be shown that this is impossible. These problems are generic to all representations of Heisenberg's form of the CCRs.

To circumvent the problems associated with unbounded operators, it is standard practice to work with the associated unitary operators, which are obtained by way of exponentiation. All unitary operators are bounded by unity. Working with bounded operators greatly simplifies matters—they are defined on the entire Hilbert space. The "Weyl unitaries" (after Hermann Weyl who first introduced them) that are associated with P and Q are by definition the following:

$$U(a) = e^{\frac{-iaP}{\hbar}} \text{ and } V(a) = e^{\frac{-iaQ}{\hbar}},$$

for any $a \in \mathbb{R}$. The Weyl form of the CCRs (for 1 degree of freedom) is then this:

$$U(a)V(b) = e^{\frac{-iab}{\hbar}} V(b)U(a)$$

for any $a, b \in \mathbb{R}$. The Weyl unitaries act on elements of $\mathcal{L}^2(\mathbb{R})$ as follows:

$$(U(a)\Psi(x)) = \Psi(x - a) \text{ and } (V(b)\Psi(x)) = e^{\frac{-ibx}{\hbar}}\Psi(x).$$

The Heisenberg form of the CCRs may be obtained by applying $\partial^2/\partial a \partial b$ to the Weyl form and then setting $a = b = 0$.[4] Finally, von Neumann proved an important theorem in 1931 that is now referred to as the Stone–von Neumann Theorem (since Stone indicated the elements of this proof in 1930). Since then, the theorem has been formulated in several different ways. The following is a contemporary formulation abstracted from Summers (2001) of the Stone–von Neumann Theorem.

Stone–von Neumann Theorem. Let $\{\tilde{U}(a)|a \in \mathbb{R}\}$, $\{\tilde{V}(b)|b \in \mathbb{R}\}$ be finite sets of weakly continuous unitary operators acting irreducibly on a separable Hilbert space $\mathcal{H}$ such that $\tilde{U}(a)\tilde{V}(b) = e^{\frac{-iab}{\hbar}}\tilde{V}(b)\tilde{U}(a)$, $\tilde{U}(a)\tilde{U}(b) = \tilde{U}(a+b)$ and $\tilde{V}(a)\tilde{V}(b) = \tilde{V}(a+b)$, then there is a Hilbert space isomorphism $W : \mathcal{H} \to \mathcal{L}^2(\mathbb{R})$ such that $W\tilde{U}(a)W^{-1} = U(a)$ and $W\tilde{V}(a)W^{-1} = V(a)$.[5]

This theorem is true for any finite system.[6] The Stone–von Neumann Theorem fails for infinite quantum systems, which typically have a continuum of unitarily *inequivalent* representations for the Weyl form of the CCRs. There are, in addition, systems having an infinite number of degrees of freedom for which the associated algebra is other than the Weyl algebra, and those algebras also have a continuum of unitarily inequivalent representations.

There are two important consequences of the Stone–von Neumann Theorem. First, the physical content of any irreducible representation of $\{\tilde{U}, \tilde{V}, \mathcal{H}\}$ is the same. In particular, wave mechanics and matrix mechanics have the same physical content and produce the same predictions for physical models. The simplest way to see their empirical equivalence is through their unitary equivalence. In quantum mechanics, states and observables correspond to special types of linear

[4] The two forms are not equivalent. For an interesting case where the Heisenberg form cannot be integrated to the Weyl form, see Reeh (1988).

[5] These sets of operators are said to act *irreducibly* on $\mathcal{H}$ if and only if the only closed subspaces of $\mathcal{H}$ that are invariant under the action of their elements are $\mathcal{H}$ and **0**. *Weak continuity* of U(t) with respect to the parameter t means that $\langle\phi|U(t)|\psi\rangle$ is a continuous function of t for each ϕ, $\psi \in \mathcal{H}$.

[6] A finite system is one that has only a finite number of degrees of freedom—i.e., the respective domains of the indices i and j are finite. This is certainly *not* to say that a separable Hilbert space must have a finite number of dimensions. The number of degrees of freedom of a system and the number of dimensions of its associated Hilbert space are distinct notions; a system with a single degree of freedom may correspond to an N-dimensional (for any positive integer $N \geq 2$) or a countably-infinite-dimensional Hilbert space. Electron spin corresponds to a single degree of freedom, and it is associated with a two-dimensional Hilbert space; whereas, the energy of a particle in an infinite square-well potential, which also corresponds to a single degree of freedom, is associated with a countably infinite-dimensional Hilbert space. A separable Hilbert space cannot have a continuum of mutually orthogonal state vectors. Dirac delta functions and plane wave functions are not elements of a separable Hilbert space. They are elements of a rigged Hilbert space, which is not a Hilbert space. Finally, an infinite system is a system having an infinite number of degrees of freedom. It is to be contrasted below with the notion of an infinite-particle system.

operators; states correspond to density operators and observables to self-adjoint operators. The expectation value for an observable A for a system that is in the state ρ is $\mathrm{Tr}(\rho A)$. The transformation of an operator O_1 from one representation to a corresponding operator O_2 in the other is effected by a unitary transformation U as $O_2 = U O_1 U^{-1}$. Let ρ_1 be a state and A_1 be an observable in one representation, and let ρ_2 and A_2 be their counterparts in the other obtained via this unitary transformation. This unitary transformation preserves expectation values—i.e., $\mathrm{Tr}(\rho_2 A_2) = \mathrm{Tr}(U\rho_1 U^{-1} U A_1 U^{-1}) = \mathrm{Tr}(\rho_1 A_1)$—since the trace operation is invariant under cyclic permutations. The empirical equivalence follows since the equality given earlier holds for all possible states and observables; the associated set of expectation values exhausts the physical content of the theory. What this means in practice is that one may freely choose to work with the most convenient representation. If a particular problem is more easily solved in matrix mechanics than wave mechanics, the unitary equivalence of the two representations guarantees that the empirical predictions would be exactly the same, if they were instead solved in wave mechanics. The situation is analogous to choosing the most convenient coordinate system.

The mathematical structure that von Neumann associated with non-relativistic, spinless quantum mechanics is a separable Hilbert space (1932).[7] It can only accommodate finite-particle systems; a finite-particle system is a system that only has a finite number of subsystems. That is to say, the domain of non-relativistic, spinless quantum mechanics is the set of finite-particle systems. This domain includes atoms, molecules, subatomic particles, and (more generally) any composite system consisting of a finite number of these systems. The same holds for spin-j systems ($j = 0, 1/2, 1, 3/2, \ldots$) and relativistic quantum mechanics. A simple example of an infinite-particle system whose set of possible vector states cannot be represented in a separable Hilbert space is an infinite lattice of spin systems. Such a system has a continuum of mutually orthonormal vector states,[8] and this means that the associated Hilbert space is *non*separable.

[7] A topological space is separable if and only if it has a countable basis. This is a consequence of separability, not a definition. By definition, a space is separable if and only if it has a countable dense subset.

[8] To see why, consider the set of real numbers in the interval [0,1] (an uncountable set). Suppose that the elements of this interval are expressed in binary notation. Consider an infinite lattice of spin systems that has a first element but no last element, and suppose that each element of the lattice is either in spin-up or spin-down with respect to the z-direction. Each distinct real number in the given interval may be associated with a distinct state of the lattice as follows. Take the sequence of 0s and 1s corresponding to a given real number in the interval, and associate the nth element of the sequence with the spin state of the nth element of the lattice by letting 1 corresponding to spin-up and 0 to spin-down (in the z-direction). It then follows that two distinct real numbers in the interval [0,1] corresponds to two mutually orthogonal lattice states—their inner product is zero due to the lattice points where they differ.

This notion of an infinite-particle system is distinct from that of a system with infinite degrees of freedom. Fock representations accommodate systems having infinite degrees of freedom; but they do not accommodate systems having an infinite number of particles (or subsystems). Other representations have been developed for treating infinite-particle systems, such as the spin lattice mentioned earlier. Instead of beginning with an associated Hilbert space, one begins with a C^*-algebra of observables that captures essential structural features of the system. A C^*-algebra is an *abstract* mathematical space that need not have an inner product, and is not a Hilbert space in general. But, it may be represented on a Hilbert space $\mathcal{H}$ by mapping it into the set of bounded operators of $\mathcal{H}$, if the mapping preserves the algebraic relations among the elements of the algebra. The associated set of operators on a Hilbert space, the range of the representation, is a *concrete* C^*-algebra. C^*-algebras and their representations are characterized more fully later.

If the system is a finite-particle system, then all representations are unitarily equivalent—i.e., for any pair there is a unitary operator that transforms one into the other. If the system is an infinite-particle system, its associated algebra has a continuum of unitarily inequivalent representations. To say that a continuum of representations is unitarily inequivalent means that for any pair of distinct representations in the continuum there is no unitary operator that transforms one into the other. Thus, algebraic quantum theory is a proper generalization of traditional quantum mechanics. It is applicable to both finite-particle and infinite-particle systems. The added generality of the algebraic framework makes possible the construction of new, viable models for a rather diverse range of physical phenomena.

There are two distinct domains of algebraic quantum theory beyond quantum mechanics. One is algebraic quantum statistical mechanics,[9] and the other is algebraic quantum field theory.[10] Infinite-particle systems are used in algebraic quantum statistical mechanics to capture rigorously key elements of thermodynamic systems. The notions of an equilibrium state, temperature, and phase transitions—such as the transition from a liquid to a gas—are three cases in point. The problem of characterizing these elements in the standard framework of quantum mechanics has proven to be intractable. The modeling of infinite-particle systems in this context crucially involves the thermodynamic limit where the volume and particle number of a gas, liquid, or solid approach infinity while the

[9] Two good introductory texts on algebraic quantum statistical mechanics are by Thirring (1980) and Sewell (1986). The definitive advanced source is the two-volume set (Brattelli and Robinson, 1979, 1981). A terrific intermediate-level source is Emch (1972).

[10] Haag (1996) and Araki (1999) provide good introductory sources on algebraic quantum field theory. Two good intermediate-level sources are the text by Horuzhy (1990) and the review article by Roberts (1990). Also, worth consulting are the more advanced treatises by Baumgärtel and Wollenberg (1995), and Baumgärtel (1995).

density remains constant. In quantum field theory, the infinite-particle system is a field; the "particles" are field effects that appear in special circumstances. In algebraic quantum field theory, a field is also a derived notion. In one type of algebraic approach, the local algebraic approach, the most fundamental entities are local observables, where the term 'local' means 'associated with a bounded spacetime region.'

2. THE ALGEBRAIC FRAMEWORK

The algebras under consideration are the (abstract) C^*-algebras and W^*-algebras, and their respective (concrete) representations as a subset of the set of bounded operators of an associated separable or nonseparable Hilbert space. These operator algebras are especially well-behaved infinite-dimensional generalizations of the algebras of $n \times n$ matrices, $n \in \mathbb{N}$.

A C^*-algebra $\mathcal{A}$ is a vector space over the field of complex numbers $\mathbb{C}$ with the following algebraic features, (a) and (b), and topological features, (c) and (d):

(a) A multiplication mapping from $\mathcal{A}$ into $\mathcal{A}$ that satisfies these three conditions for all $A, B, C \in \mathcal{A}$ and $\lambda \in \mathbb{C}$: $A(B + C) = AB + AC$, $A(BC) = (AB)C$, $A(\lambda B) = \lambda(AB)$.
(b) A conjugation mapping * from $\mathcal{A}$ into $\mathcal{A}$ that satisfies these three conditions for all $A, B \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{C}$: $(A^*)^* = A$, $(AB)^* = B^*A^*$, $(\lambda A + \mu B)^* = \bar{\lambda}A^* + \bar{\mu}B^*$.
(c) A norm $\|.\|$ that satisfies $\|AB\| \leq \|A\|\|B\|$ and $\|A^*A\| = \|A\|^2$ for all $A, B \in \mathcal{A}$.
(d) Completeness with respect to the norm topology, the topology given by the metric induced by the norm.

Suppose now that $\mathcal{A}$ is a C^*-algebra. The complete set of bounded linear functionals on $\mathcal{A}$ is by definition its *dual*, $\mathcal{A}^*$.[11] It can be shown that $\mathcal{A}^*$ is a Banach space.[12] A *state* is a positive linear functional of unit norm. The complete set of states in $\mathcal{A}^*$ is denoted as $\mathcal{A}^{*+}$. It is often described as a convex subset of the unit ball of $\mathcal{A}^*$, which means that $\mathcal{A}^{*+}$ is closed under convex sums; ω is as a *convex sum* of $\omega_1, \omega_2 \in \mathcal{A}^{*+}$, if $\omega = \lambda\omega_1 + (1 - \lambda)\omega_2$ for some $0 \leq \lambda \leq 1$. The *extremal elements* of $\mathcal{A}^{*+}$ are those that can only be trivially expressed as a convex sum, meaning that the sum exists only if $\lambda = 0$ or $\lambda = 1$; they are referred to as the *pure states* on $\mathcal{A}$.

[11] An index of notation is included towards the end this paper for the reader's convenience.
[12] A Banach space is a normed vector space that is complete with respect to the metric induced by the norm.

By definition, a W^*-algebra is a C^*-algebra that is the dual of a Banach space.[13] So, if $\mathcal{R}$ is a W^*-algebra, there is a (unique) Banach space $\mathcal{R}_*$ such that $\mathcal{R} = (\mathcal{R}_*)^*$ is referred to as the *predual* of $\mathcal{R}$. It consists of the complete set of normal states on $\mathcal{R}$. A linear functional ρ on a W^*-algebra $\mathcal{R}$ is said to be *normal* if and only if $\rho(\sup_\alpha \mathrm{T}_\alpha) = \sup_\alpha \rho(\mathrm{T}_\alpha)$ for every uniformly bounded increasing directed set $\{T_\alpha\}$ of positive elements of $\mathcal{R}$ (Sakai, 1971, Definition 1.13.1). In the concrete setting, a normal state is a positive trace-class-1 Hilbert-space operator.[14]

Suppose now that $\mathcal{A}$ is a C^*-algebra. Since $\mathcal{A}^*$ is a Banach space, as noted earlier, the *bidual* of $\mathcal{A}$ (i.e., the dual of $\mathcal{A}^*$) is well defined; it is denoted as $\mathcal{A}^{**}$, where $\mathcal{A}^{**} = (\mathcal{A}^*)^*$. Thus, $\mathcal{A}^{**}$ is a W^*-algebra by the earlier definition. $\mathcal{A}^{**}$ plays a key role later, since it is isometrically isomorphic with the universal enveloping von Neumann algebra of $\mathcal{A}$, which is the von Neumann algebra generated by the universal representation of $\mathcal{A}$. The universal representation of $\mathcal{A}$ involves the entire set of representations of $\mathcal{A}$, which indicates that $\mathcal{A}^{**}$ is a very large space.[15]

As already mentioned, a representation of a C^*-algebra $\mathcal{A}$ is a map π from $\mathcal{A}$ into the set of bounded operators $\mathcal{B}(\mathcal{H}_\pi)$ of an associated Hilbert space $\mathcal{H}_\pi$ that preserves the algebraic relations between the elements of $\mathcal{A}$. The resulting concrete C^*-algebra, the image of $\mathcal{A}$ in H_π, is denoted as $\pi(\mathcal{A})$. To say that π preserves the algebraic relations between the elements of $\mathcal{A}$ means that the following conditions are satisfied for any $A, B \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$:

- $\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$,
- $\pi(AB) = \pi(A)\pi(B)$, and
- $\pi(A^*) = \pi(A)^*$.

$\pi(A)$ denotes the operator in $\mathcal{H}_\pi$ that corresponds to $A \in \mathcal{A}$. A representation π is *faithful* if and only if $\ker\pi = \{0\}$, meaning that the kernel of π is the zero element of $\mathcal{A}$. By definition, $\ker\pi \equiv \{A \in \mathcal{A} | \pi(A) = 0\}$. So, a faithful representation is one that only maps the zero observable in $\mathcal{A}$ to the zero operator in $\mathcal{H}_\pi$, and this may be expressed by saying that the kernel of the representation (the set of zero elements) is *trivial*. It is standard practice in algebraic quantum theory to restrict attention to faithful representations; i.e., π is required to be injective.

The standard procedure for generating representations of a C^*-algebra $\mathcal{A}$ is the GNS construction (so called since Gelfand, Naimark, and Segal first formulated it). They proved that for every $\rho \in \mathcal{A}^{*+}$, one may construct a representation π_ρ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_{\pi_\rho}$ in such a way that there is a vector $\phi_\rho \in \mathcal{H}_{\pi_\rho}$ that

[13] One example of a C^*-algebra that is *not* a W^*-algebra is the algebra of continuous complex-valued functions on a compact set. It can be shown that this algebra is not the dual of a Banach space.

[14] If $T \in \mathcal{B}(\mathcal{H})$ and T is positive (i.e., $\langle\psi|T\psi\rangle \geq 0$ for all $\psi \in \mathcal{H}$) then it is of *trace class*, provided that $\mathrm{Tr}(T) < +\infty$; it is *trace-class-1*, if (in addition) $\mathrm{Tr}(T) = 1$. Normal states are discussed in more detail later when von Neumann algebras are introduced.

[15] Some proponents of the algebraic approach regard it as too large. This issue is not addressed below.

satisfies these two conditions for all $A \in \mathcal{A}$:

$$\rho(A) = \langle \phi_\rho | \pi_\rho(A)\phi_\rho \rangle \quad \text{and} \quad \mathcal{H}_{\pi_\rho} = \overline{\{\pi_\rho(A)\phi_\rho | A \in \mathcal{A}\}}.$$

Each triple $\langle \mathcal{H}_{\pi_\rho}, \pi_\rho, \phi_\rho \rangle$ is unique up to unitary equivalence.[16]

A representation π is said to be *irreducible* if and only if the only closed subspaces of $\mathcal{H}_\pi$ that are invariant under the action of the elements of $\pi(\mathcal{A})$ are $\mathcal{H}_\pi$ and $\mathbf{0}$. It can be shown that the representation associated with an element $\rho \in \mathcal{A}^{*+}$ is irreducible if and only if ρ is a pure state. An operator on $\mathcal{H}_\pi$ is a *density operator*, if it is a trace-class-1 operator. An element of $\rho \in \mathcal{A}^{*+}$ is said to be π*-normal*, if it can be represented as a density operator ω_ρ in $\mathcal{H}_\pi$; that is, if π maps ρ to a density operator $\omega_\rho \in \mathcal{H}_\pi$, then ρ is a π-normal state of $\mathcal{A}^{*+}$. The set of all π-normal states form a norm closed convex subset of $\mathcal{A}^{*+}$ that is called the *folium* $\mathcal{F}_\pi$ associated with the representation π.

One very useful algebraic structure for algebraic quantum theory is the notion of a von Neumann algebra. By definition, a *von Neumann algebra* is a weakly closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. To say that a subset $\mathcal{R}$ of $\mathcal{B}(\mathcal{H})$ is *weakly closed* (i.e., that $\mathcal{R}$ is *closed in the weak operator topology*) means that any sequence $\{T_n\}$ of elements of $\mathcal{R}$ converges to another element $T \in \mathcal{R}$ in the sense that $\langle \phi | T_n | \psi \rangle \to \langle \phi | T | \psi \rangle$ for all $\phi, \psi \in \mathcal{H}$. A von Neumann algebra may also be defined algebraically using the notion of a bicommutant. Suppose $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$. $\mathcal{R}$'s *commutant* is $\mathcal{R}' = \{y \in \mathcal{B}(H) : \forall x \in \mathcal{R}, xy = yx\}$, and $\mathcal{R}$'s bicommutant is $\mathcal{R}'' = (\mathcal{R}')'$. By definition, $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ is a *von Neumann algebra* if and only if $\mathcal{R} = \mathcal{R}''$. It can be shown that these two definitions of a von Neumann algebra are equivalent.

A von Neumann algebra may be generated by a representation π of a C^*-algebra $\mathcal{A}$ in two ways, one topological and the other algebraic. The topological way is to close $\pi(\mathcal{A})$ in the weak operator topology, and the resulting von Neumann algebra is denoted as $\pi(\mathcal{A})^-$. The algebraic way is to take the bicommutant of $\pi(\mathcal{A})$, and the resulting von Neumann algebra is denoted as $\pi(\mathcal{A})''$. $\pi(\mathcal{A})''$ is a von Neumann algebra, since it can be shown that $\pi(\mathcal{A})'''' = \pi(\mathcal{A})''$. It turns out that these two methods of generating von Neumann algebras are equivalent, meaning that $\pi(\mathcal{A})''$ and $\pi(\mathcal{A})^-$ are identical. $\pi(\mathcal{A})''$ is used later to denote the von Neumann algebra generated by the representation π of $\mathcal{A}$. Finally, it is worth noting that a W^*-algebra is an abstract von Neumann algebra—see Theorem 1.16.7 of Sakai (1970, pp. 41–42).

3. PHYSICAL EQUIVALENCE AND FELL'S THEOREM

As noted earlier, the Stone–von Neumann Theorem fails for infinite non-relativistic, spinless quantum systems. Such systems typically have a continuum

[16] For details, see Bratteli and Robinson (1979, 1981, Section 2.3.3).

of unitarily *inequivalent* representations for the Weyl form of the CCRs or, more generally, for systems that are characterized by some other algebra of bounded observables. Proponents of algebraic quantum field theory regarded the inequivalent representations as an overabundance. Unitary equivalence implies exact empirical equivalence. If representations are unitarily inequivalent, one is left to wonder whether they are at least empirically equivalent in some sense. If a reasonable notion of empirical equivalence cannot be found, then it will be necessary to introduce criteria for representation selection. To resolve this selection problem, Haag and Kastler (1964) introduced the notion of physical equivalence, which is related to Fell's (1960) notion of weak equivalence, and a theorem proved by Fell, which they reformulate in terms of physical equivalence.

Let π_1 and π_2 be two representations of a C^*-algebra $\mathcal{A}$. A topology may be defined on $\mathcal{A}^*$ called the weak*-topology in which the weak*-neighborhoods of a state ϕ are indexed by a finite subset of $\mathcal{A}$ and positive real ε. Let $\mathcal{B} \equiv \{A_i\}_{i=1}^k$ with $\mathcal{B} \subset \mathcal{A}$ and $k \in \mathbb{N}$, and let $N(\phi, \mathcal{B}, \varepsilon)$ denote a generic weak*-neighborhood of ϕ. A state ψ is said to be in $N(\phi, \mathcal{B}, \varepsilon)$ if and only if $|\phi(A_i) - \psi(A_i)| < \varepsilon$ for $1 \leq i \leq k$. Using the weak*-topology and related notions characterized earlier, Haag and Kastler introduce the following definition. Two representations π_1, π_2 of a C^*-algebra $\mathcal{A}$ are *physically equivalent* if and only if for every π_1-normal state ϕ (i.e., $\phi \in \mathcal{F}_{\pi_1}$) and every weak*-neighborhood $N(\phi, \mathcal{B}, \varepsilon)$ of ϕ there is a π_2-normal state ψ such that ψ is in $N(\phi, \mathcal{B}, \varepsilon)$, and *vice versa* (1964, p. 851). They justify the use of the term "physical" by interpreting ε as the maximum experimental error associated with members of $\mathcal{B}$, and by noting that all actual experiments only involve a finite number of observables that are each measured with a finite precision. Here is a more explicit characterization of the notion of physical equivalence.

Physical Equivalence. Two representations π_1, π_2 of a C^*-algebra $\mathcal{A}$ are *physically equivalent* iff (if and only if) for every $\phi \in \mathcal{F}_{\pi_1}$, finite set $\{A_i\}_{i=1}^k \subset \mathcal{A}$, and real $\varepsilon > 0$, there is a $\psi \in \mathcal{F}_{\pi_2}$ such that $|\phi(A_i) - \psi(A_i)| < \varepsilon$ for $1 \leq i \leq k$.[17]

Haag and Kastler go on to say that what gives this definition of physical equivalence its incisive, penetrating quality is (their version of) Fell's Theorem.

Fell's Theorem. All faithful representations of a C^*-algebra of observables are physically equivalent (Fell, 1960).[18]

[17] Emch gives a fuller characterization of physical equivalence than Haag and Kastler in Theorem 7 and the associated lemma (Emch, 1972, pp. 106–107). In the paragraph following Theorem 7, Emch implies (in his final remark) that the characterization of physical equivalence of Haag and Kastler corresponds to a rephrasing of one pair of conditions of the theorem—he means conditions (iii) and (iv) of the associated lemma.

[18] Fell's notion of weak equivalence is purely mathematical, whereas Haag and Kastler's notion of physical equivalence is a *physical interpretation* of weak equivalence. As noted earlier, the epsilons are interpreted as experimental errors and the restriction to a finite subset of $\mathcal{A}$ is interpreted as a limitation on the number of measurements that can actually be carried out in a given experiment.

As noted earlier, algebraic quantum theorists are only interested in faithful representations. This means that Fell's Theorem effectively resolves the selection problem. No actual experiment can serve to distinguish between unitarily inequivalent representations that are faithful.[19] So, one can simply choose any convenient faithful representation without worrying about there being any detectable physical differences in predictions with respect to any other representation.

4. LOCAL ALGEBRAIC QUANTUM FIELD THEORY

In local algebraic quantum field theory, each open region $\mathcal{O} \in \mathcal{M}$ ($\mathcal{M}$ is Minkowski spacetime) is associated with a set $\mathcal{A}(\mathcal{O})$ of elements of a C^*-algebra, the local observables in $\mathcal{O}$. The regions $\mathcal{O} \in \mathcal{M}$ are often taken to be double-cones, nonempty intersections of the interiors of a forward and a backward light cone.[20] $\mathcal{A}_{\text{loc}} \equiv \overline{\bigcup_{\mathcal{O}\in\mathcal{M}} \mathcal{A}(\mathcal{O})}$ is the set of quasi-local algebra over $\mathcal{M}$, where the overhead bar denotes the closure of the associated set in the norm topology. $\mathcal{A}_{\text{loc}}$ is assumed to satisfy a set fundamental physical conditions, known as "axioms of local structure." Three key axioms of local structure are isotony, locality, covariance:

$$\begin{array}{ll} \text{Isotony}: & \text{If}\,\mathcal{O}_1 \subset \mathcal{O}_2 \text{ then } \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2), \\ \text{Locality}: & \text{If}\,O_1 \times \mathcal{O}_2 \text{ then } [\mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2)] = 0, \\ \text{Covariance}: & \text{If}\,g \in \mathcal{P}_+^\uparrow \text{ and } A \in \mathcal{A}(\mathcal{O}), \alpha_g[\mathcal{A}(\mathcal{O})] = \mathcal{A}(g[\mathcal{O}])\text{and } U(g)AU(g)^{-1} = \alpha_g[A]. \end{array}$$

The meaning of isotony is evident. Locality says that if two spacetime regions are space-like separated, then any element from the algebra associated with one region must commute with any element from the algebra associated with the other. Covariance says that each element g of the restricted Poincare group $\mathcal{P}_+^\uparrow$ (one of four disjoint classes of the Poincare group) may be represented as an automorphism α_g of the algebra $\mathcal{A}(\mathcal{O})$. These are the primary conditions, though additional conditions are often specified. It is not necessary to state these or to elaborate further on the ones already specified, since they are not the focus of the discussion that follows.

According to Haag and Kastler, $\mathcal{A}_{\text{loc}}$ captures all of the physically relevant features of the associated quantum field. But there are larger algebras that contain $\mathcal{A}_{\text{loc}}$ that have physically significant elements that are not in $\mathcal{A}_{\text{loc}}$. Two such

[19] There is an important consequence of Fell's Theorem that is noted in Haag and Kastler (1964, p. 852). It involves simple C^*-algebras, which are frequently the type of quasi-local algebra used in algebraic quantum field theory. All representations of a simple C^*-algebra are faithful. So, it follows from Fell's Theorem that all representations of a simple C^*-algebra are physically equivalent.

[20] Some quantum field models are formulated in bounded spatial regions, as in algebraic quantum statistical mechanics, as opposed to bounded spacetime regions. Free-field models satisfying the Klein–Gordon equation can be formulated in this manner (these are the so-called "fixed-time" or "equal-time" formulations of the free-field model), but this is not the case for arbitrary Wightman fields and generalized free-fields (Horuzhy, 1990, pp. 241 ff.).

algebras are considered later: von Neumann algebras generated by representations of $\mathcal{A}_{\text{loc}}$ and the universal enveloping von Neumann algebra generated by the universal representation of $\mathcal{A}_{\text{loc}}$. The generated elements of these algebras (i.e., the elements of these algebras that are not in $\mathcal{A}_{\text{loc}}$) are referred to as "global observables." One important subset of the set of global observables is the set of classical observables. Such observables commute with each other as well as with all other observables of the system. This commutation condition is a necessary condition for being a classical observable, but it is not clear whether it is sufficient for being such. Some writers define the classical observables more narrowly as the set of observables at infinity, which are physical quantities that are outside of any bounded spacetime region. We adopt this more restrictive definition for the purposes of this paper.[21]

The upshot is that Haag and Kastler regard global observables as physically insignificant. They do so for seemingly operational reasons; i.e., on the grounds that it is not possible to measure such quantities—this would especially seem to be the case for observables at infinity, as the name suggests. But, developments in quasi-local algebraic quantum statistical mechanics (and also in quantum mechanics in phase space) seem to cast doubt on their position.

5. QUASI-LOCAL ALGEBRAIC QUANTUM STATISTICAL MECHANICS

In quasi-local algebraic quantum statistical mechanics, the quasi-local C^*-algebra of observables is defined as follows. For each finite open sphere $s \in \mathcal{E}$ ($\mathcal{E}$ is Euclidean space) and time $t \in \mathbb{R}$ there is an associated C^*-algebra of observables $\mathcal{A}(s, t)$ whose elements are local observables in s at t. The quasi-local algebra at t is $\mathcal{A}_{\text{loc}}(t) \equiv \overline{\bigcup_{s\in\mathcal{E}} \mathcal{A}(s, t)}$. The quasi-local algebra is $\mathcal{A}_{\text{loc}} \equiv \overline{\bigcup_{t\in\mathbb{R}} \mathcal{A}(t)}$. $\mathcal{A}_{\text{loc}}$ is assumed to satisfy certain fundamental physical conditions that are counterparts to the ones given earlier for algebraic quantum field theory.[22] The "loc" subscript is suppressed in the remainder of this section and in the next—the context indicates well enough when it is intended.

[21] The more restrictive definition is suggested by Hepp (1972, p. 241), and by Primas (1983, pp. 187–189). Their reasons for adopting this more restrictive identification are unclear. One argument that could be given for it is that classical observables are often treated as parameters (i.e., real scalars). If so, then they would belong to the observables at infinity, since they belong to every algebra defined on each finite region and on regions that are causally disjoint from these regions. In any case, the issue deserves further analysis in a future paper.

[22] Isotony is exactly analogous to that in algebraic quantum field theory, and it involves spatial regions instead of spacetime regions. Locality corresponds to commutativity of operators that are associated with disjoint spatial regions rather than space-like separated regions of spacetime. Covariance is often associated with a continuous one parameter group of *-automorphisms of the C^*-algebra of observables rather than the Poincare group.

The results in quasi-local algebraic quantum statistical mechanics mentioned earlier have to do with equilibrium states, which are known as KMS states—the name is a tribute to the mathematical physicists who discovered them (Kubo, Martin, Schwinger). For a given thermodynamic system, meaning an infinite quantum system with an associated quasi-local algebra $\mathcal{A}$, there is a KMS state associated with each value β, the inverse temperature of the system. It will suffice to focus on systems having a finite inverse temperature $0 < \beta < +\infty$.[23] A KMS state ϕ that corresponds to a pure phase of an infinite system at a finite temperature β induces a representation π_ϕ of $\mathcal{A}$ via the GNS construction. The associated von Neumann algebra $\pi_\phi(\mathcal{A})''$ is a factor.[24] A *factor* is a von Neumann algebra with a trivial center. The *center* of a von Neumann algebra is by definition the intersection of the algebra with its commutant. It is said to be *trivial* if its elements consist of scalar multiples of the identity element.

Takesaki (1970) showed that two factor representations π_ϕ and π_Ψ are disjoint provided that one of them is a type III factor and they correspond respectively to inverse temperature values β, γ such that $\beta \neq \gamma$.[25] A representation π of a C^*-algebra $\mathcal{A}$ is a *factor representation* if and only if the associated von Neumann algebra $\pi(\mathcal{A})''$ is a factor. Two representations of a C^*-algebra are *disjoint* if no subrepresentation of one is equivalent to a subrepresentation of the other. A *subrepresentation* of a representation π of a C^*-algebra $\mathcal{A}$ on Hilbert space $\mathcal{H}$ is a representation that is obtained by restricting π to a non-zero subspace of $\mathcal{H}$ that is invariant under $\pi(\mathcal{A})$.

6. PHYSICAL EQUIVALENCE REVISITED

To connect the considerations from quasi-local algebraic quantum statistical mechanics given earlier with those concerning unitarily inequivalent representations and physical equivalence, two simple theorems are derived. They are consequences of theorems that have been proven elsewhere—see references that follow. The proofs of those theorems are not repeated here. The first theorems is stated as follows:

Theorem 1. *Two factor representations π_ϕ and π_ψ are disjoint if and only if they are not quasi-equivalent.*

[23] It is theoretically possible to allow for β to have the values 0, $+\infty$, and even negative values. These possibilities are not relevant for what follows.

[24] It turns out that the algebra is a type III factor. Murray and von Neumann categorized factors into three mutually exclusive and exhaustive types—see Chapter 1 of Sunder (1987) for a brief review of their classificatory scheme.

[25] In 1980, Müller-Herold proved a similar result for chemical potential based on Takesaki's result.

Two representations π_ϕ, π_ψ of a C^*-algebra $\mathcal{A}$ are *quasi-equivalent* if and only if there is an isomorphism $\alpha : \pi_\phi(\mathcal{A})'' \to \pi_\psi(\mathcal{A})''$ such that $\pi_\psi(\mathcal{A}) = \alpha(\pi_\phi(A))$ for each $A \in \mathcal{A}$. In this context, an isomorphism is a bijection which preserves the algebraic operations, which includes the * operation.

Theorem 1 is a consequence of Corollary 10.3.4 and Proposition 10.3.12 in Kadison and Ringrose (1986, pp. 737–742). By Proposition 10.3.12(ii), two factor representations are disjoint, if they are not quasi-equivalent. It remains to show that they are disjoint only if they are not quasi-equivalent. Suppose that π_ϕ and π_ψ are disjoint factor representations. Let $\pi_{\hat{\phi}}$ and $\pi_{\hat{\psi}}$ be a subrepresentation of each, respectively. By Corollary 10.3.4(i), two representations are disjoint iff they have no quasi-equivalent subrepresentations. So, $\pi_{\hat{\phi}}$ and $\pi_{\hat{\psi}}$ are not quasi-equivalent. By Proposition 10.3.12(i), a representation is a factor representation iff it is quasi-equivalent to each of its subrepresentations. So, π_ϕ is quasi-equivalent to $\pi_{\hat{\phi}}$, and π_ψ is quasi-equivalent to $\pi_{\hat{\Psi}}$. Assume for *reductio* that π_Φ and π_Ψ are quasi-equivalent. It then follows that there are three *-automorphisms α, β, γ (corresponding respectively to the three quasi-equivalencies) such that $\pi_\phi(A) = \alpha(\pi_{\hat{\phi}}(A)), \pi_\psi(A) = \beta(\pi_{\hat{\psi}}(A))$, and $\pi_\psi(A) = \gamma(\pi_\phi(A))$ for each $A \in \mathcal{A}$. So, there must be a *-automorphism δ such that $\pi_{\hat{\phi}}(A) = \delta(\pi_{\hat{\psi}}(A))$ for each $A \in \mathcal{A}$, where $\delta = \alpha^{-1}\gamma^{-1}\beta$. But this means that $\pi_{\hat{\phi}}$ and $\pi_{\hat{\psi}}$ are quasi-equivalent, and this contradicts a previous claim.[26]

The other theorem that is needed to connect the considerations from quasi-local algebraic quantum statistical mechanics given earlier with those concerning inequivalent representations is an immediate consequence of Theorem 12 in Emch (1972, p. 124), and the theorem is stated as follows.

Theorem 2. *Two representations π_ϕ and π_ψ of a C^*-algebra $\mathcal{A}$ are quasi-equivalent if and only if $\tilde{\pi}_\phi$ and $\tilde{\pi}_\psi$ are physically equivalent.*

The representations π_ϕ and π_ψ can be uniquely extended to the universal enveloping von Neumann algebra $\pi^u(\mathcal{A})''$. Their extensions are denoted as $\tilde{\pi}_\phi$ and $\tilde{\pi}_\psi$, respectively. $\pi^u(\mathcal{A})$ is the universal representation of the algebra $\mathcal{A}$, which is defined as the direct sum of all representations formed from the states in $\mathcal{A}^{*+}_{\cdot}\tilde{\pi}_\phi$ and $\tilde{\pi}_\psi$ are equivalent to $\pi_\phi(\mathcal{A})''$ and $\pi_\psi(\mathcal{A})''$, respectively (Emch, 1972, p. 122). The universal representation of $\mathcal{A}$ and the universal enveloping von Neumann algebra are explained in more detail in the next section.

The upshot for the issues under consideration is the following consequence of Theorems 1 and 2, namely the following theorem.

[26] Thanks go to Rob Clifton for providing us with this version of the proof. One of the authors of this paper (T.L.) had a different but equally effective proof of this theorem involving additional concepts.

Theorem 3. *If π_ϕ and π_ψ are disjoint factor representations of a C^*-algebra $\mathcal{A}$, then their von Neumann extensions $\tilde{\pi}_\phi$ and $\tilde{\pi}_\psi$ are not physically equivalent.*

Two disjoint factor representations of $\mathcal{A}$ are not quasi-equivalent by Theorem 1, and by Theorem 2 this means that their von Neumann extensions are not physically equivalent. More concretely, if ϕ and Ψ are KMS states associated with inverse temperature values β, γ with $0 < \beta < +\infty, 0 < \gamma < +\infty$, and $\beta \neq \gamma$, then $\tilde{\pi}_\phi$ and $\tilde{\pi}_\psi$ are not physically equivalent.[27]

It is necessary to explain what it means to say that $\tilde{\pi}_\phi$ and $\tilde{\pi}_\psi$ are not physically equivalent. The matter is subtler than it first appears. It means the following:

> $\tilde{\pi}_\phi$ and $\tilde{\pi}_\psi$ are physically inequivalent if there is a $\phi' \in \mathcal{F}_{\pi_\phi}$, a finite set $\{A_i\}_{i=1}^k \subset \pi^u(\mathcal{A})''$, and an associated set of positive reals $\{\varepsilon_i\}_{i=1}^k$, such that for all $\psi' \in \mathcal{F}_{\pi_\psi}$, $|\phi'(A_i) - \psi'(A)_i| \geq \varepsilon_i$ for some $i (1 \leq i \leq k)$.

Note that the notion of physical equivalence mentioned earlier involves finite subsets of $\pi^u(\mathcal{A})''$ and not merely finite subsets of $\mathcal{A}$. This is an important distinction, since $\tilde{\pi}_\phi$ and $\tilde{\pi}_\psi$ are physically equivalent with respect to any finite subset $\{A_i\}_{i=1}^k \subset \mathcal{A} : \tilde{\pi}_\phi$ and $\tilde{\pi}_\psi$ are extensions of π_ϕ and π_ψ, and the latter are physically equivalent by Fell's Theorem.

From Theorem 7 and the associated lemma in Emch (1972, pp. 106–107), it follows that $\tilde{\pi}_\phi$ and $\tilde{\pi}_\psi$ are physically inequivalent if and only if $\ker \tilde{\pi}_\phi \neq \ker \tilde{\pi}_\psi$. Given the earlier discussion, it follows that if π_ϕ and π_ψ are disjoint factor representations, then $\ker \tilde{\pi}_\phi \neq \ker \tilde{\pi}_\psi$. If $\ker \tilde{\pi}_\phi \neq \ker \tilde{\pi}_\psi$, then there are two possible cases: one representation is faithful and the other is not, or both representations are not faithful. In either case, there is an $A \in \pi^u(\mathcal{A})''$ such that $A \neq 0$ and either $\tilde{\pi}_\phi(A) = 0$ or $\tilde{\pi}_\psi(A) = 0$. We may suppose without loss of generality that $\tilde{\pi}_\phi(A) \neq 0$ and $\tilde{\pi}_\psi(A) = 0$. It follows that for any density matrix $\rho_{\tilde{\pi}_\phi}$ in the folium of $\tilde{\pi}_\phi$ and any density matrix $\rho_{\tilde{\pi}_\psi}$ in the folium of $\tilde{\pi}_\psi$ that $\rho_{\tilde{\pi}_\phi}(A) \neq \rho_{\tilde{\pi}_\psi}(A) = 0$. That is to say, no density matrix $\rho_{\tilde{\pi}_\phi}$ in the folium of $\tilde{\pi}_\phi$ can approximate any density matrix $\rho_{\tilde{\pi}_\psi}$ in the folium of $\tilde{\pi}_\psi$ for the observable $A \in \pi^u(\mathcal{A})''$. Thus, there will always be some global observable $A \in \pi^u(\mathcal{A})''$ that will physically distinguish between any pair of disjoint factor representations.

The earlier considerations serve to cast doubt on the claim that only faithful representations are physically significant and on the claim that all of the physical content of a theory is contained in the algebra of local or quasi-local observables. Haag and Kastler regard local observables as the only physically significant ones for the purposes of their local quantum physics project. Classical observables, a subset of the set of observables at infinity, are treated as parameters that index superselection sectors. The Takesaki and Müller-Herold results

[27] A similar result holds for chemical potential.

came after the Haag and Kastler paper. Those results were used here to show that even if one were to relegate classical observables (or, more generally, observables at infinity) to the status of parameters, there would still be at least one global observable that would serve to physically distinguish between unitarily inequivalent representations of the quasi-local algebras. Such observables are elements of the von Neumann algebra generated by representations of the quasi-local algebra of observables: they are not elements of the quasi-local algebra. This means that the policy of focusing solely on local and quasi-local observables and the policy of invoking the concept of physical equivalence to deal with the unitarily inequivalent representations of the local or quasi-local algebra have serious limitations.

7. OBSERVABLES AT INFINITY

Aside from the earlier considerations, there are other reasons for regarding Haag and Kastler's position on observables at infinity as very dubious. It is certainly not clear that the observables at infinity are unmeasurable in principle. Indeed, it certainly seems that it is possible to measure temperature and chemical potential. Although these quantities arise in the context of quasi-local algebraic quantum statistical mechanics, it is reasonable to suppose that such quantities will have counterparts in thermal quantum field theory. So, it is worth indicating how such observables may be treated in a unified manner along with the local, quasi-local, and global observables.

Let $\mathcal{A}$ be a C^*-algebra of quasi-local observables in $\mathcal{S}$.[28] Let $\mathcal{O}$ be an open bounded region of $\mathcal{S}$ and $\mathcal{O}^\perp$ the region of $\mathcal{S}$ that is (causally) disjoint from $\mathcal{O}$. The C^*-algebra of observables in $\mathcal{O}^\perp$ is $\mathcal{A}(\mathcal{O}^\perp) \subset \mathcal{A}$. If π is a representation of $\mathcal{A}$, the von Neumann algebra in $\mathcal{O}^\perp$ that is generated by π is $\mathcal{B}_\pi(\mathcal{O}) \equiv \pi(\mathcal{A}(\mathcal{O}^\perp))''$, the algebra of observables associated with $\mathcal{O}^\perp$. The algebra of observables at infinity is then defined as $\mathcal{B}_\pi \equiv \bigcap_{\mathcal{O}\in\mathcal{S}} \mathcal{B}_\pi(\mathcal{O})$. $\mathcal{B}_\pi$ is often characterized as a subset of the center of $\pi(\mathcal{A})''$. It is worth explaining the significance of this claim. The center of $\pi(\mathcal{A})''$ is usually denoted as $\mathcal{Z}_\pi(\mathcal{A})$ and (as mentioned earlier) it is defined as follows: $\mathcal{Z}_\pi(\mathcal{A}) \equiv \pi(\mathcal{A})' \cap \pi(\mathcal{A})''$. It is well known that if π is a representation of $\mathcal{A}$, then $\mathfrak{B}_\pi \subset \mathcal{Z}_\pi(\mathcal{A})$.[29] Elements of $\mathfrak{B}_\pi$ are not elements of $\pi(\mathcal{A})$, but they are elements of $\pi^u(\mathcal{A})''$; so, they are global observables. Presumably, classical observables are interpreted as elements of $\mathfrak{B}_\pi$, since the latter commute with each other as well as with all other observables of the system. Mutual commutativity is a characteristic feature of the elements of the set

[28] $\mathcal{S} = \mathcal{E}$ for algebraic quantum statistical mechanics. Usually, $\mathcal{S} = \mathcal{M}$ for algebraic quantum field theory, though sometimes $\mathcal{S} = \mathcal{E}$ is used instead.

[29] See Proposition 2.1 and the associated proof on page 196 of Lanford and Ruelle (1969).

of classical observables; though, it should be added that there is no reason to think that all the elements of $\mathfrak{B}_\pi$ correspond to some classical observable or other.[30]

As noted earlier, a *factor* is by definition a von Neumann algebra with a trivial center, meaning that the elements of its center are just scalar multiples of the identity element. KMS states, equilibrium states corresponding to definite temperature values, are factor representations. This means that the temperature observable in a factor representation is merely a multiple of the identity; that is to say, it is merely a parameter value. This suggests that a larger representation is necessary to represent classical observables such as temperature as nontrivial elements of the algebra of observables that are associated with the system. One representation that may serve this purpose is the *universal representation* π^u of the algebra $\mathcal{A}$ of quasi-local observables. By definition it is the direct sum of all GNS representations over all states—i.e., it is defined as $\pi^u \equiv \bigoplus_{\rho\in\mathcal{A}^{*+}} \pi_\rho$. Its associated concrete C^*-algebra is $\pi^u(\mathcal{A})$, where $\pi^u(\mathcal{A}) \equiv \bigoplus_{\rho\in\mathcal{A}^{*+}} \pi_\rho(\mathcal{A})$, and it is defined on the direct sum of Hilbert spaces $\mathcal{H}_{\pi^u} \equiv \bigoplus_{\rho\in\mathcal{A}^{*+}} \mathcal{H}_{\pi_\rho}$. Classical observables may be interpreted as nontrivial global elements of $\pi^u(\mathcal{A})''$, the universal enveloping von Neumann algebra of $\pi^u(\mathcal{A})$, as suggested in Müller-Herold (1980).[31] Müller-Herold also suggests that a properly chosen subalgebra of $\pi^u(\mathcal{A})''$ may be more appropriate, thereby hinting that $\pi^u(\mathcal{A})''$ may perhaps be too large. For systems having a finite inverse temperature $0 < \beta < +\infty$, one could take the representation corresponding to a direct integral of (type III) factor representations each associated with a different temperature value. The temperature observable is then a global element of the bidual of the algebra corresponding to this representation.[32]

8. CONCLUSION

One could try to defend the philosophy of local quantum physics (that all the physically relevant features of a system are contained in the

[30] Sometimes a classical observable of a system is characterized as an observable for which the system has a precise value at all times. Surely this characterization is too restrictive a notion for the macroscopic observables that are characterized in classical statistical mechanics. A system has a definite temperature if and only if it is in an equilibrium state, and certainly there are states that are far from and others that are near to equilibrium.

[31] From what is given earlier in Section 2, it follows that $\pi^u(\mathcal{A})^- = \pi^u(\mathcal{A})'' \cong \mathcal{A}^{**}$, where $\pi^u(\mathcal{A})^-$ is the closure $\pi^u(\mathcal{A})$ in the weak operator topology, $\mathcal{A}^{**}$ (a W^*-algebra) is the bidual of $\mathcal{A}$, and the relation "$\alpha \cong \beta$" means that α isometrically isomorphic to β.

[32] Positive developments along these lines have already been put forward in Amann and Müller-Herold (1986) and in Amann (1987).

quasi-local C^*-algebra of observables) by arguing that all that is really ever measured are local or quasi-local approximations to these global "idealized" quantities.[33] This is a worthy suggestion, and this matter is briefly addressed here.

The observables in question are the global observables. Global observables are the elements of the universal enveloping von Neumann algebra $\pi^u(\mathcal{A})''$ generated by a representation π of a C^*-algebra $\mathcal{A}$ that do not correspond to elements of $\pi(\mathcal{A})$. Among these are the observables at infinity, which include the classical observables. Mathematically, these observables arise due to a change in topology. The quasi-local algebra is closed in the norm topology, the associated enveloping von Neumann algebra is closed in the weak operator topology, and many sequences of observables that do not converge in the norm topology do so in the weak operator topology. For any of the global observables in question, it is not possible to find a sequences of observables in the quasi-local algebra that approximate it in the norm topology as closely as one would like (for any epsilon). Of course, the philosophy of local quantum physics only requires that the approximation be as close as one would like for all practical purposes. Suppose for the sake of argument that one can specify a lower limit on approximation for experimental purposes. There is still the question as to whether the global observables in question that are actually measurable can be so approximated once the lower limit is set. There is no *a priori* reason for thinking that this is so. The only way that the defense can be carried out is on a case-by-case basis. One case of special interest is the temperature observable. Perhaps the proponents of this philosophy can specify the element of the quasi-local algebra that suitably approximates this observable. Temperature measurements are made, but one must know that the system is in an equilibrium state to say that the value obtained in a local measurement corresponds to the temperature of the system. That is to say, a global determination must be made before the local measurement has the appropriate significance.

Rather than defend the philosophy of local quantum physics, we propose two alternative approaches: the first involves understanding what it means from a physical point of view (both epistemic and ontic) to adopt a change in topology.[34] The second is to build directly a non-local C^*-algebra of observables. These approaches give rise to issues that are certainly worth addressing in more detail, but this must be done elsewhere.

[33] Rob Clifton suggested this possibility to us.

[34] For more on the point of view advocated here, see Primas (1998) and Atmanspacher and Kronz (1999).

INDEX OF NOTATION

$\mathcal{E}$ Euclidean space

$\mathcal{M}$ Minkowski spacetime

$\mathcal{O}$ an open bounded region of some global framework $\mathcal{S}$ (typically $\mathcal{E}$ or $\mathcal{M}$)
- $\mathcal{E}$ is used in quasi-local algebraic quantum statistical mechanics
- $\mathcal{M}$ is usually used in algebraic quantum field theory; sometimes $\mathcal{E}$ is used

$\mathcal{A}$ a C^*-algebra of observables

$\mathcal{A}(\mathcal{O})$ a C^*-algebra of observables that is associated with a region $\mathcal{O}$ of $\mathcal{S}$

A_{loc} a C^*-algebra of quasi-local observables that is associated with $\mathcal{S}$
- $\mathcal{A}_{\text{loc}} \equiv \overline{\bigcup_{\mathcal{O}\in\mathcal{S}} \mathcal{A}(\mathcal{O})}$, the bar denotes closure with respect to the norm topology
- the "loc" subscript is suppressed in the text when there is no risk of confusion

$\mathcal{A}^*$ the dual of $\mathcal{A}$—it is the complete set of continuous linear functionals on $\mathcal{A}$

$\mathcal{A}^{*+}$ the state space of $\mathcal{A}$—it is the complete set of positive unit-norm elements of $\mathcal{A}^*$

$\mathcal{A}^{**}$ the bidual (the dual of the dual) of $\mathcal{A}$; i.e., $\mathcal{A}^{**} \equiv (\mathcal{A}^*)^*$

$\mathcal{H}$ a Hilbert space

$\mathcal{B}(\mathcal{H})$ the set of bounded operators on $\mathcal{H}$

π a representation of $\mathcal{A}$ in $\mathcal{H}_\pi$, the Hilbert space associated with π
- π is a mapping of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_\pi)$ that faithfully preserves the algebraic relations between elements of $\mathcal{A}$ (including the * operation)

$\mathcal{H}_\pi$ the Hilbert space associated with the representation π

$\mathcal{F}_\pi$ the folium associated with the representation π, which is the set of π-normal states in $\mathcal{A}^{*+}$; a state is π-normal iff it can be represented as a density operator in $\mathcal{H}_\pi$

$\pi(\mathcal{A})$ a concrete C^*-algebra generated by a representation π of a C^*-algebra $\mathcal{A}$

$\pi(\mathcal{A})'$ the commutant of $\pi(\mathcal{A})$

$\pi(\mathcal{A})''$ the bicommutant of $\pi(\mathcal{A})$, which is the von Neumann algebra generated by π

$\mathcal{Z}_\pi(\mathcal{A})$ the center of $\pi(\mathcal{A})''$, where (by definition) $\mathcal{Z}_\pi(\mathcal{A}) \equiv \pi(\mathcal{A})' \bigcap \pi(\mathcal{A})''$

π^u the universal representation of $\mathcal{A}$, where $\pi^u \equiv \bigoplus_{\rho\in\mathcal{A}^{*+}} \pi_\rho$

$\mathcal{H}_{\pi^u}$ the Hilbert space associated with π^u, where $\mathcal{H}_{\pi^u} \equiv \bigoplus_{\rho\in\mathcal{A}^{*+}} \mathcal{H}_{\pi\rho}$

$\pi^u(\mathcal{A})$ the C^*-algebra that corresponds to the universal representation of $\mathcal{A}$, where $\pi^u(\mathcal{A}) \equiv \bigoplus_{\rho\in\mathcal{A}^{*+}} \pi_\rho(\mathcal{A})$

$\pi^u(\mathcal{A})''$ the universal enveloping von Neumann algebra of $\pi^u(\mathcal{A})$
- $\pi^u(\mathcal{A})''$ is isometrically isomorphic with $\mathcal{A}^{**}$

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